JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 26, No. 1, January–February 2003

Flight Motion of a Continuously Elastic Finned Missile

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The motion of elastic finned projectiles has been analyzed by various approximate theories. The correct equations of small-amplitude motion are derived for a symmetric missile. The aerodynamic and elastic symmetries are used to allow the use of complex variables to describe the lateral motion in a nonrotating coordinate system. Although the resulting equations are both ordinary and partial differential equations, frequencies and damping rates of free oscillations are obtained from an ordinary differential equation with boundary conditions. The induced drag coefficients for each mode are computed as functions of the missile's elasticity. Equations for a permanently deformed bent missile are derived, and an ordinary differential equation for the forced motion of a bent missile is obtained. Sample calculations for a finned projectile with fineness ratio of 20 show resonant motion at the aerodynamic frequency, as well as at each elastic frequency. The nonlinear roll moment associated with a bent missile is computed and the location of possible spin–yaw lock-in is determined.

Nomenclature

$A(x)$ = local cross-sectional area
$c_{D\,\mathrm{bp}}$ = base pressure coefficient
$c_D(x)$ = drag force distribution function
$c_{fj}(x)$ = aerodynamic force distributions functions
d = rod diameter
E = Young's modulus
F = complex transverse aerodynamic force, $F_y + iF_z$
$\boldsymbol{F}$ = aerodynamic force exerted on missile, (F_x, F_y, F_z)
$\hat{F}_a$ = complex aerodynamic shear force on rod at x
$\hat{F}_d$ = complex beam damping shear force on rod at x
$\hat{F}_e$ = complex beam elastic shear force on beam at x
g_t = I_t/md^2
g_x = I_x/md^2
g_1 = $\rho V^2 S/2$
g_2 = g_1/md
g_3 = $\rho V S d/2m$
g_4 = $g_2 L/\omega_0^2$
$\boldsymbol{H}$ = angular momentum of missile, (H_x, H_y, H_Z)
I = area moment of rod,

$$(d)^4 \iint y^2 \,\mathrm{d}y\,\mathrm{d}z = (d)^4 \iint z^2 \,\mathrm{d}y\,\mathrm{d}z$$

I_d = moment of inertia of disk normal to symmetry axis
I_t = transverse moment of inertia of rigid projectile
I_x = axial moment of inertia of rigid projectile
$I\{z\}$ = imaginary part of z
$2I_d$ = moment of inertia of disk about symmetry axis
$\hat{k}$ = beam damping coefficient
L = rod length/rod diameter
M = complex transverse aerodynamic moment, $M_y + iM_z$
$\boldsymbol{M}$ = aerodynamic moment exerted on missile, (M_x, M_y, M_z)
M_e = complex elastic moment on rod at x, $M_{ey} + iM_{ez}$
m = projectile mass
p = angular velocity component along symmetry axis of central disk; projectile spin
p_d = angular velocity component along symmetry axis of disk
Q = complex transverse angular velocity of central disk, $q + ir$
$R\{z\}$ = real part of z
S = $\pi d^2/4$
V = magnitude of projectile velocity
$\boldsymbol{V}$ = velocity of central disk
$\boldsymbol{V}_c$ = velocity of center of mass
$\boldsymbol{V}_d$ = velocity of disk
X = central disk location, 0
x_1, x_2 = location of beam ends
x_{01}, x_{23} = dimensionless length of fore and aft aerodynamic extensions
α = angle of attack of central disk
β = angle of sideslip of central disk
Γ = complex cant of disk, $\partial\delta/\partial x$
δ = complex lateral displacement of disk relative to missile's c.m., $\delta_E - \delta_c$
$\boldsymbol{\delta}$ = dimensionless location of disk relative to missile's c.m., (x, δ_y, δ_z)
δ_c = complex lateral location of missile's c.m., $\delta_{cy} + \delta_{cz}$
$\boldsymbol{\delta}_c$ = dimensionless location of c.m., $(0, \delta_{cy}, \delta_{cz})$
δ_E = complex lateral displacement of disk, $\delta_{Ey} + i\delta_{Ez}$
$\boldsymbol{\delta}_E$ = dimensionless location of disk, $(x, \delta_{Ey}, \delta_{Ez})$
η = complex angle of attack of disk
θ = pitch angle of central disk with respect to Earth-fixed axes
λ_k = damping of kth mode
$\hat{\lambda}_K$ = parameter in Eq. (47)
ξ = complex angle of attack of central disk, $\beta + i\alpha$
ρ = air density
σ = ω_1/ω_R
ϕ = roll angle
$\dot{\phi}_k$ = frequency of kth mode
ψ = yaw angle of central disk with respect to Earth-fixed axes
$\boldsymbol{\Omega}$ = angular velocity of nonspinning elastic coordinates, $(0, q, r)$
$\boldsymbol{\omega}$ = angular velocity of central disk, (p, q, r)
ω_R = rigid projectile zero-spin frequency
ω_1 = lowest elastic frequency of beam in vacuum
ω_0^{-2} = $EI(L/md)$

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Subscripts

B	=	parameter for bent projectile
BF	=	bent fin parameter
E	=	elastic coordinate parameter
T	=	trim motion parameter

Superscripts

^	=	parameter for bent missile
~	=	elastic parameter for bent missile

Introduction

MODERN antitank kinetic energy projectiles have become very long-finned missiles carrying very dense metal rods. Because these projectiles have to resist very large aerodynamic loads due to their high-velocity flight at sea level, designers have been concerned about the possibility of aeroelastic deformations.[1–5] There is some evidence that a small number of these projectiles have been forced to spin at rates close to the their lowest elastic frequency and have been subject to large inelastic deformations. According to linear theory, the spin should symtotically approach its design steady-state value. Nonlinear spin moments, however, can cause the spin to lock-in at the spin associated with large resonant trim motion. In Refs. 3 and 5, special solutions have been developed that showed spin lock-in at the lowest elastic frequency.

Mikhail[3] used an incorrect expression for the angular momentum and assumed erroneously that a spinning elastic missile could perform planar flexing motion and a planar pitching motion in a coordinate system rotating with the missile. His numerical calculations, however, showed examples of spin lock-in when fin damage produces a roll inducing moment sufficient to cause a steady-state spin greater than the lowest elastic frequency and initial spin was zero. Heddadj et al.[4] approximated an elastic missile by two rigid bodies elastically connected and studied the transient motion of these bodies, but did not consider spin lock-in.

Murphy and Mermagen[5] approximated the continuously elastic missile by three rigid bodies connected by two massless elastic beams and derived differential equations of motion for this three-body system. The equations showed that it is impossible to cause spin lock-in by roll inducing moment and zero initial spin alone. In Ref. 6, it was shown that the combination of an offset center of mass and a trim pitch moment creates a nonlinear roll moment that can produce spin lock-in at the aerodynamic frequency. The rear beam of the three-body theoretical model was given an inelastic lateral deformation, and spin lock-in at the lowest elastic frequency was observed.

Because the approximate equations in Ref. 3 are far from being theoretically correct, those results can be dismissed. The equations in Ref. 5 are valid, but the use of the three-body model is a major simplification of the actual physical problem. It is the purpose of this paper to derive the correct equations for the small-amplitude motion of a continuously elastic symmetric projectile. This derivation exploits the missile's aerodynamic and elastic symmetry by describing the transverse motion with complex variables and using a nonrotating coordinate system with its much simpler equations of motion.

Although the resulting equations form a combination of ordinary and partial differential equations containing integrals of the elastic deformation, free oscillations and constant-spin trim motion of a bent missile can be computed from fourth-order ordinary differential equations. Induced drag for various free oscillations are deduced as quadratic functions of their amplitude. Quadratic roll moments associated with a bent missile are formulated for trim motion. Equilibrium spins that can cause spin-pitch lock-in are located. Numerical integration of the complete set of partial and ordinary differential equations for varying spin would be necessary to verify the actual occurrence of lock-in. The three-body theory of Ref. 5 would, however, be a good indicator of this lock-in behavior.

Coordinate System

If we consider a symmetric missile with no elastic flexing, we can define a nonspinning coordinate system with origin at the rigid missile's center of mass and X axis directed along the axis of symmetry. The Z axis is perpendicular to the X axis and initially downward pointing, whereas the Y axis is determined by the right-hand rule.

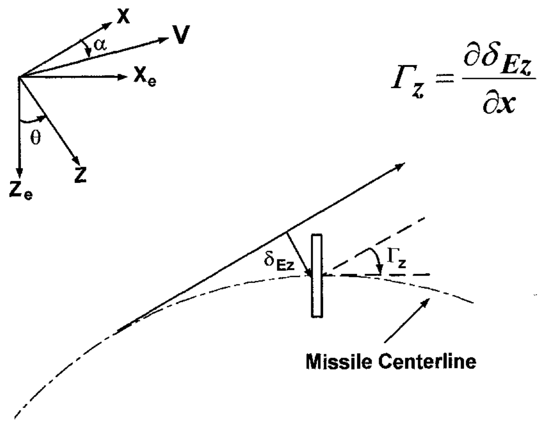

Fig. 1 ***XZ* coordinates of cross-sectional disk.**

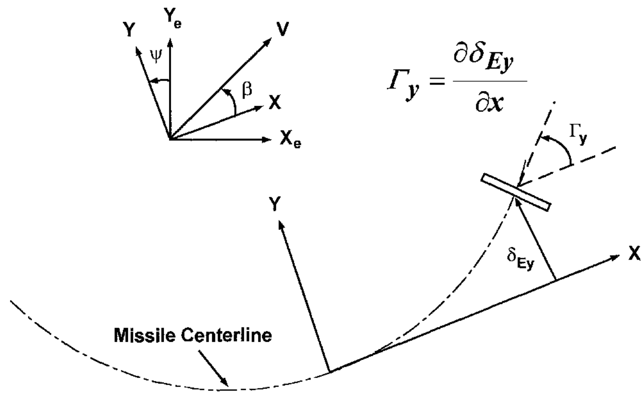

Fig. 2 ***XY* coordinates of cross-sectional disk.**

The elastic missile is assumed to consist of a very heavy elastic circular rod of diameter d and fineness ratio L embedded in a very light symmetric aerodynamic outer structure that may be longer than the rod. We will conceptually slice the missile into a large number of thin disks perpendicular to the X axis with thickness dx. For the elastic rod, an elastic nonspinning coordinate system is defined so that its origin at the center of the central disk, and the X axis is tangent to the rod at this point.

This elastic coordinate system can be specified with respect to Earth-fixed $X_eY_eZ_e$ coordinates by pitch angle θ and yaw angle ψ. When the rod flexes, the disks shift laterally perpendicular to the X axis and cant to be perpendicular to the centerline of the disks. Figures 1 and 2 show the coordinates of a cross-sectional disk. It will be convenient to use nondimensional coordinates: $x = X/d$, $y = Y/d$, and $z = Z/d$.

The angular velocity of the central disk in these nonspinning coordinates is defined to be the vector

$$\boldsymbol{\omega} = (p, q, r) \cong (\dot{\phi}, \dot{\theta}, \dot{\psi}) \tag{1}$$

Because the coordinate system pitches and yaws with the missile but does not spin, its angular velocity vector is

$$\boldsymbol{\Omega} = (0, q, r) \tag{2}$$

A dimensionless vector specifies the location of the displaced center of a disk at x with respect to the origin of the elastic coordinate system:

$$\boldsymbol{\delta}_E(x, t) = [x, \delta_{Ey}(x, t), \delta_{Ez}(x, t)] \tag{3}$$

The elastic coordinate system has been defined to move, pitch, and yaw so that

$$\delta_{Ey}(0, t) = \delta_{Ez}(0, t) = 0 \tag{4}$$

$$\frac{\partial \delta_{Ey}(0,t)}{\partial x} = \frac{\partial \delta_{Ez}(0,t)}{\partial x} = 0 \tag{5}$$

For small angles, the velocity of the central disk can be defined to be

$$\boldsymbol{V} = (V, V\alpha, V\beta) \tag{6}$$

The mass of the very light aerodynamic structure will be neglected, and the linear density of the projectile is, therefore, $\rho_1 = m/Ld$. The lateral location of the center of mass of the flexing elastic missile is

$$\boldsymbol{\delta}_c = (0, \delta_{cy}, \delta_{cz}) \tag{7}$$

where

$$\delta_{cj} = (1/L)\int_{x_1}^{x_2} \delta_{Ej}\,\mathrm{d}x$$

The time derivative of any vector in the elastic coordinate system, $\boldsymbol{R} = (r_x, r_y, r_z)$, can be computed by the usual formula,

$$\frac{\partial \boldsymbol{R}}{\partial t} = (\dot{r}_x, \dot{r}_y, \dot{r}_z) + \boldsymbol{\Omega} \times \boldsymbol{R} \tag{8}$$

where

$$\dot{r}_j = \frac{\partial r_j}{\partial t}$$

The velocity of the center of mass can be expressed by the relation

$$\boldsymbol{V}_c = \boldsymbol{V} + (0, \dot{\delta}_{cy}, \dot{\delta}_{cz})d + \boldsymbol{\Omega} \times \boldsymbol{\delta}_c d \tag{9}$$

Finally the velocity of the any disk has a relation similar to that for $\boldsymbol{V}_c$,

$$\boldsymbol{V}_d = \boldsymbol{V} + (x, \dot{\delta}_{Ey}, \dot{\delta}_{Ez})d + \boldsymbol{\Omega} \times \boldsymbol{\delta}_E d$$

$$= V_c + (x, \dot{\delta}_y, \dot{\delta}_z)d + \boldsymbol{\Omega} \times \boldsymbol{\delta} d \tag{10}$$

where

$$\boldsymbol{\delta} = \boldsymbol{\delta}_d - \boldsymbol{\delta}_c = (x, \delta_y, \delta_z)$$

Center of Mass Motion

If the aerodynamic force acting on the missile is denoted by $\boldsymbol{F} = (F_x, F_y, F_z)$, the center of mass motion of the projectile must satisfy the usual Newtonian equation:

$$m\frac{\mathrm{d}\boldsymbol{V}_c}{\mathrm{d}t} = \boldsymbol{F} \tag{11}$$

The linearized x component of this equation becomes

$$m\dot{V} = F_x \tag{12}$$

The rotational symmetry of the missile can be exploited by multiplying the z component equation by i and adding it to the y component equation,

$$m(\dot{\xi} + \dot{V}\xi + \ddot{\delta}_c - iVQ) = F \tag{13}$$

where

$$\xi = \beta + i\alpha, \qquad Q = q + ir$$

$$\delta_c = \delta_{cy} + \delta_{cz}, \qquad F = F_y + iF_z$$

Angular Motion of the Projectile

Each disk is canted at the angle Γ_y in the XY plane and Γ_z in the XZ plane. The moment of inertia of the disk normal to its symmetry axis is $I_d = (d^3/16)\rho_1\,\mathrm{d}x$, and the moment of inertia along the symmetry axis is $2I_d$. In a coordinate system aligned with the symmetry axis of the disk, its angular velocity is $(p_d, q + \dot{\Gamma}_z, r - \dot{\Gamma}_y)$ and its angular momentum is $[2I_d p_d, I_d(q + \dot{\Gamma}_z), I_d(r - \dot{\Gamma}_y)]$.

Because spin–yaw lock-in is a nonlinear quadratic process, it is necessary to retain quadratic terms in the spin equation. The quadratic approximation to the unit vector along the symmetry axis of a disk is

$$\boldsymbol{e}_d = \left[1 - \left(\tfrac{1}{2}\right)\left(\Gamma_y^2 + \Gamma_z^2\right), \Gamma_y, \Gamma_z\right] \tag{14}$$

The quadratic approximation to the angular velocity along this axis is

$$p_d = \boldsymbol{e}_d \circ \boldsymbol{\omega} = p(1 - \Gamma\bar{\Gamma}/2) + q\Gamma_y + r\Gamma_z \tag{15}$$

In the elastic coordinate system, the angular momentum of a disk with respect to its center of mass is

$$\boldsymbol{h}_d(\rho_1 d)\,\mathrm{d}x = (h_x, h_y, h_z) \tag{16}$$

where

$$h_x = [2p_d(1 - \Gamma\bar{\Gamma}/2) + \Gamma_y(q + \dot{\Gamma}_z) + \Gamma_z(r - \dot{\Gamma}_y)]I_d$$

$$h_y = (q + \dot{\Gamma}_z + 2p\Gamma_y)I_d, \qquad h_z = (r - \dot{\Gamma}_y + 2p\Gamma_z)I_d$$

According to Newtonian mechanics, the derivative of the angular momentum of a system with respect to its center of mass is equal to the external moment with respect to its center of mass. The angular momentum of the elastic missile with respect to its center of mass is

$$\boldsymbol{H} = \int_{x_1}^{x_2} [\boldsymbol{h}_d + (\boldsymbol{\delta}_E - \boldsymbol{\delta}_c) \times (\boldsymbol{V}_d - \boldsymbol{V}_c)](\rho_1 d)\,\mathrm{d}x \tag{17}$$

Equations (9) and (10) can be employed to yield the three components of the angular momentum vector:

$$H_x = I_x p + md^2 R\{J_7 - Q\bar{J}_6\} \tag{18}$$

$$H_y + iH_z = I_t Q + imd^2(\dot{J}_6 + J_8) \tag{19}$$

where J_j are defined in Appendix A.

If the missile is assumed to fly at constant pitch angle and its deformed shape is rotating as a rigid body ($Q = 0$, $\dot{\delta} = ip\delta$), the axial angular momentum is

$$H_x = \hat{I}_x p \tag{20}$$

where

$$\hat{I}_x = I_x + \left(\frac{md^2}{L}\right)\int_{x_1}^{x_2}\left[\frac{|\delta|^2 - |\Gamma|^2}{16}\right]\mathrm{d}x$$

Mikhail[3] computed $|\delta(t)|^2$ for a flexing missile, neglected the $|\Gamma|^2$ term, and assumed that Eq. (20) was valid for the axial angular momentum for any motion. This assumption is clearly incompatible with Eq. (18).

The derivative of the angular momentum vector specified by Eq. (17) is equal to the aerodynamic moment:

$$\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = \boldsymbol{M} = (M_x, M_y, M_z) \tag{21}$$

For a slender missile, $\dot{J}_8$ can be neglected in comparison with $\ddot{J}_6$.

The differential equations for the missile's angular motion are

$$I_x\dot{p} + md^2 R\{\dot{J}_7 - \dot{Q}\bar{J}_6\} = M_x \tag{22}$$

$$I_t\dot{Q} - ipI_x Q + imd^2\ddot{J}_6 = M \tag{23}$$

Aerodynamic Force

The aerodynamic force acting on a rigid projectile is usually assumed to be proportional to ξ, Q, and their derivatives. For an elastic missile, a local aerodynamic force needs to be defined in a similar manner. The x coordinate of a point on the rod axis lies between $x_1 = -L/2$ and $x_2 = L/2$. The aerodynamic structure incasing the rod can have a nose windshield of length $x_{23}d$, and its fins could extend beyond the end of the rod by the distance $x_{01}d$. Thus, the aerodynamic force is exerted from $x_0 = x_1 - x_{01}$ to $x_3 = x_2 + x_{23}$. If we assume the aerodynamic structure to be quite rigid beyond the ends of the rod,

$$\delta_E(x,t) = \delta_E(x_1,t) + (x - x_1)\Gamma(x_1,t) \qquad x_0 \le x \le x_1 \tag{24}$$

$$\delta_E(x,t) = \delta_E(x_2,t) + (x - x_2)\Gamma(x_2,t) \qquad x_2 \le x \le x_3 \tag{25}$$

The transverse aerodynamic force exerted on a disk located at x will be assumed to have the form appropriate to a pointed slender body.[7] Thus, the force will be proportional to the local angle of attack and its modified time derivative:

$$\mathrm{d}F = \frac{\partial F}{\partial x}\,\mathrm{d}x = -g_1\left[c_{f1}\eta + c_{f2}(\dot\eta + iQ)\left(\frac{d}{V}\right)\right]\mathrm{d}x \tag{26}$$

where $F = F_y + iF_z$, $c_{fn}(x)$ are two normal force distribution coefficients per axial length, and η is the local complex angle of attack. The distribution functions derived for Munk's airship theory[8] (also see Ref. 7) are, for example,

$$c_{f1} = -2\frac{\mathrm{d}A}{\mathrm{d}x}S^{-1} \qquad c_{f2} = 2AS^{-1} \tag{27}$$

A general pointed slender body relationship between c_{f1} and c_{f2} is

$$c_{f2} = -\int_{x_3}^{x} c_{f1}\,\mathrm{d}x \tag{28}$$

In Ref. 9, computer programs for calculating c_{f1} for finned missiles are given.

The complex angle between the missile's centerline described by $\delta_E(x,t)$ and the X axis will be denoted as Γ and it is the spatial derivative of δ_E:

$$\Gamma \equiv \frac{\partial \delta_E}{\partial x} = \frac{\partial \delta}{\partial x} \tag{29}$$

The local angle of attack is the angle between the velocity vector and the surface of the missile, which is assumed to be parallel to the centerline,

$$\eta = \xi + (\dot\delta_E - ixQ)d/V - \Gamma \tag{30}$$

Equation (30) can be differentiated to yield a relation for $\dot\eta$. For the aerodynamic frequency and lower flexing frequencies, the angular acceleration terms in $\ddot\delta_E$ and $\dot Q$ are small and can be neglected:

$$\dot\eta = \dot\xi - \dot\Gamma \tag{31}$$

The total transverse aerodynamic force acting on the missile can be obtained by integrating Eq. (26) from x_0 to x_3:

$$F = -g_1[C_{N\alpha}\xi + C_{N\dot\alpha}(\dot\xi d/V) + iC_{Nq}(Qd/V) - J_1(t) - \dot J_2(t)(d/V)] \tag{32}$$

where C_N are listed in Appendix B and $J_j(t)$ are listed in Appendix A. The axial aerodynamic force is directed along the X axis and can be approximated by use of an axial drag coefficient per length and a base pressure drag coefficient $c_{D\,\mathrm{bp}}$

$$F_x = -g_1 C_{DX} \cong -g_1 C_D \tag{33}$$

where

$$C_{DX} = \int_{x_0}^{x_3} c_D(x)\,\mathrm{d}x + c_{D\,\mathrm{bp}}$$

Equations (12) and (33) yield the well-known drag equation

$$m\dot V = -g_1 C_D \tag{34}$$

Equations (13) and (32) can be combined to yield a simple relation between Q and $\dot\xi$. Small terms involving C_{Nq} and $C_{N\dot\alpha}$ have been neglected,

$$(V/d)(\dot\xi - iQ) = -g_2 C_{L\alpha}\xi + N \tag{35}$$

where $C_{L\alpha} = C_{N\alpha} - C_D$ and N is defined in Appendix A.

Aerodynamic Moment

The transverse aerodynamic moment exerted on the missile is computed with respect to the center of mass and has contributions from both the transverse aerodynamic force and the axial aerodynamic force:

$$\mathrm{d}M = -i(g_1 d)\lfloor[c_{f1}\eta + c_{f2}(\dot\eta d/V)]x - c_D\delta\rfloor\,\mathrm{d}x \tag{36}$$

where $M = M_y + iM_z$. The total transverse aerodynamic moment acting on the projectile can be obtained by integrating Eq. (36) and adding the moment due to base pressure,

$$M = -i(g_1 d)\lfloor C_{M\alpha}\xi + C_{M\dot\alpha}(\dot\xi d/V) + iC_{Mq}(Qd/V) - J_3(t) - \dot J_4(t)(d/V) - J_5(t)\rfloor \tag{37}$$

where the C_{Mj} are given Appendix B.

Equations (23), (35), and (37) can now be combined to give a simple second-order differential equation ξ and various integrals of δ:

$$I_t\ddot\xi + (a_2 - ipI_x)\dot\xi + (a_1 + ipa_3)\xi = J_E + J_N + md^2\ddot J_6 \tag{38}$$

where

$$a_1 = -(g_1 d)\lfloor C_{M\alpha} + g_3 C_{L\alpha}C_{Mq}\rfloor \qquad a_3 = -\left(g_1 d^2/V\right)g_x C_{L\alpha}$$

$$a_2 = -\left(g_1 d^2/V\right)(C_{Mq} + C_{M\dot\alpha} - g_t C_{L\alpha})$$

and J_E and J_N are defined in Appendix A.

For a typical missile g_3 is less than 10^{-4}, and the second term in a_1 can be neglected as well as a similar term in J_N. As a general rule, iQ can replace $\dot\xi$ in all of the aerodynamic force and aerodynamic moment expressions. The rigid-projectile frequencies and damping rates can be computed from Eq. (38) for the right-side set equal to zero. Very good approximations for the frequencies are[10]

$$\dot\phi_{kR} = \left\lfloor pI_x \pm \sqrt{(pI_x)^2 + 4I_t a_1}\right\rfloor\Big/2I_t \cong \pm\omega_R + (1 \pm pI_x/4I_t\omega_R)pI_x/2I_t \qquad k = 1,2 \tag{39}$$

where $\omega_R = \sqrt{(a_1/I_t)}$ is the rigid-projectile zero-spin frequency.

The aerodynamic roll moment is the x component of the aerodynamic moment about the projectile's center of mass. The linear roll moment coefficient for a rigid finned projectile is usually expressed in terms of a roll damping coefficient and a steady-state spin,

$$(C_\ell)_{\mathrm{linear}} = C_{\ell p}(p - p_{\mathrm{ss}})(d/V) \tag{40}$$

The steady-state spin is usually determined by a differential canting of the fins caused either intentionally by the designer or unintentionally by damage to the projectile.

The roll moment of one of the projectile's thin disks is the sum of the linear roll moment it has as part of a projectile and the quadratic roll moment induced by the transverse aerodynamic force acting on its lateral displacement relative to the missile's c.m. If we retain only the dominant c_{f1} term in Eq. (26), the quadratic roll moment has the form

$$(\mathrm{d}M_x)_{\mathrm{quadratic}} = \lfloor(\mathrm{d}F_z)\delta_y - (\mathrm{d}F_y)\delta_z\rfloor = (g_1 d)c_{f1}R\{i[(\xi - \Gamma) + (\dot\delta_E - ixQ)(d/V)]\bar\delta\}\,\mathrm{d}x \tag{41}$$

The total aerodynamic roll moment acting on the projectile, therefore, is

$$M_x = (g_1 d)[(C_\ell)_{\text{linear}} + R\{J_9(t)\}] \tag{42}$$

where J_9 is defined in Appendix A.

The nonlinear roll moment from Eq. (42) can be placed in the spin equation (22) to yield

$$g_x \dot{p} + R\{\dot{J}_7 - \dot{Q}\bar{J}_6 - g_2 J_9\} = g_2 (C_\ell)_{\text{linear}} \tag{43}$$

This equation is nonlinear due to the retention of all three quadratic terms.

Flexing Motion of the Projectile

Classical beam theory for a circular beam assumes the beam to be slender and the elastic moment exerted on the right of a cross-sectional disk, M_e, to be proportional to the second derivative of the displacement of the beam. Although this is usually stated in a coordinate system rotating with the beam, the rotational symmetry of the beam allows us to state the proportionality in the nonspinning coordinates:

$$-iM_e = \left(\frac{EI}{d}\right)\frac{\partial^2 \delta_E}{\partial x^2} \tag{44}$$

where $M_e = M_{ey} + iM_{ez}$. A similar proportionality for the elastic shear force exerted on the right can be obtained from beam theory:

$$\hat{F}_e = -\frac{\partial\left[(EI/d^2)\left(\partial^2\delta_E/\partial x^2\right)\right]}{\partial x} \tag{45}$$

For a homogenous circular rod with constant diameter, EI is constant, and the net elastic shear force on a disk is

$$\therefore \frac{\partial \hat{F}_e}{\partial x}\,\mathrm{d}x = -\left(\frac{md}{L}\right)\omega_0^2 \frac{\partial^4 \delta_E}{\partial x^4}\,\mathrm{d}x \tag{46}$$

Note $\hat{\omega}_0$ has the dimensions of a frequency and appears in the expression for the flexure frequencies of a free–free beam with no aerodynamic force present:

$$\omega_K = (\hat{\lambda}_K/L)^2 \omega_0 \tag{47}$$

where $\hat{\lambda}_K = 4.730, 7.853, 10.996, 14.137$, etc.

A very common beam damping assumption is the Kelvin–Voight model (see Ref. 11) that requires the beam damping shear force to be proportional to the time derivative of the elastic shear force. This time derivative, however, must be taken in rod-fixed coordinates that rotate with the rod:

$$\frac{\partial \hat{F}_d}{\partial x}\,\mathrm{d}x = -\left(\frac{md}{L}\right)\omega_0^2 \hat{k} c_{d4}\,\mathrm{d}x \tag{48}$$

where

$$c_{dr} = 2\omega_1^{-1}\left[\frac{\partial}{\partial t}\left(\frac{\partial^r \delta_E}{\partial x^r} e^{-i\phi}\right)\right]e^{i\phi} = 2\omega_1^{-1}\frac{\partial^r}{\partial x^r}(\dot{\delta}_E - ip\delta_E)$$

and c_{dr} is scaled by $2\omega_1^{-1}$ so that $\hat{k} = 1$ corresponds to critical damping of the first elastic mode.

The aerodynamic shear force is a combination of the aerodynamic force distributions $c_{f1}(x)$ and $c_{f2}(x)$ appearing in Eq. (26) and the drag-induced moment distribution $c_D(x)$ appearing in Eq. (33). According to Geradin and Rixen,[12] the net aerodynamic shear force on a disk is

$$\frac{\partial \hat{F}_a}{\partial x}\,\mathrm{d}x = \left[\frac{\partial F}{\partial x} - g_1\frac{\partial(c_D\delta)}{\partial x}\right]\mathrm{d}x \tag{49}$$

Although c_{f1} and c_{f2} can have a finite number of jump discontinuities, c_D must have only a finite number of jump discontinuities in its derivative.

The acceleration of a disk can be obtained by differentiating Eq. (10) and using Eq. (12). The linear expression for the lateral components of this acceleration is

$$a_{dy} + ia_{dz} = F/m + (\ddot{\delta} - ix\dot{Q})\,\mathrm{d} \tag{50}$$

The equation of motion for each disk is, therefore,

$$\left(\rho_1 d^2\right)\left(\frac{\ddot{\delta} - ix\dot{Q} + F}{md}\right) = \frac{\partial}{\partial x}(\hat{F}_e + \hat{F}_d + \hat{F}_a) \tag{51}$$

The aerodynamic force distribution and its integral, $\partial F/\partial x$ and F, are available from Eqs. (26) and (32), and the other quantities are provided by Eqs. (35), (46), (48), and (49):

$$\frac{\partial^2\delta_E}{\partial t^2} + \omega_0^2\frac{\partial^4\delta_E}{\partial x^4} + \omega_0^2\hat{k}c_{d4} - g_2 L\Bigg[c_{f1}\Gamma + (c_{f2}\dot{\Gamma} - c_{f1}\dot{\delta}_E)\left(\frac{d}{V}\right) - \frac{\partial(c_D\delta)}{\partial x}\Bigg] = +E_1\xi + E_2\dot{\xi} + ix\dot{Q} - N \tag{52}$$

where $E_j(x)$ are defined in Appendix A. Equation (52) is based on the usual assumptions of neglecting the canting inertia of the spinning disks and any shear deformation. These small effects can be calculated if desired.[12]

The boundary conditions at x_1 are determined by the aerodynamic force and moment exerted on the overhanging fins. The elastic force conditions have an additional term from the drag-induced moment distribution[12]:

$$\frac{\partial^3\delta_E(x_1,t)}{\partial x^3} + \hat{k}c_{d3}(x_1,t) + g_4 c_D(x_1)\delta(x_1,t) = -\left(\frac{g_4}{g_1}\right)\int_{x_0}^{x_1}\frac{\partial F}{\partial x}\,\mathrm{d}x = -g_4 f_1 \tag{53}$$

$$\frac{\partial^2\delta_E(x_1,t)}{\partial x^2} + \hat{k}c_{d2}(x_1,t) = -\left(\frac{g_4}{g_1}\right)\int_{x_0}^{x_1}\left[(x-x_1)\frac{\partial F}{\partial x} - \delta\frac{\partial F_x}{\partial x}\right]\mathrm{d}x = g_4 m_1 \tag{54}$$

where f_j, m_j are defined in Appendix A. Similarly the boundary conditions at x_2 are set by the aerodynamic force and moment exerted on the nose extension

$$\frac{\partial^3\delta_E(x_2,t)}{\partial x^3} + \hat{k}c_{d3}(x_2,t) + g_4 c_D(x_2)\delta(x_2,t) = \left(\frac{g_4}{g_1}\right)\int_{x_2}^{x_3}\frac{\partial F}{\partial x}\,\mathrm{d}x = g_4 f_2 \tag{55}$$

$$\frac{\partial^2\delta_E(x_2,t)}{\partial x^2} + \hat{k}c_{d2}(x_2,t) = \left(\frac{g_4}{g_1}\right)\int_{x_2}^{x_3}\left[(x-x_2)\frac{\partial F}{\partial x} - \delta\frac{\partial F_x}{\partial x}\right]\mathrm{d}x = -g_4 m_2 \tag{56}$$

If beam damping and drag are neglected and the aerodynamic structure does not extend beyond the rod ($\hat{k} = c_D = x_{01} = x_{23} = 0$), the boundary conditions reduce to those for a free–free beam:

$$\frac{\partial^3\delta_E}{\partial x^3} = \frac{\partial^2\delta_E}{\partial x^2} = 0 \qquad x = x_1, x_2 \tag{57}$$

A fourth-order partial differential equation usually has four boundary conditions. In this case, however, Eqs. (4) and (5) represent two more conditions at $x = 0$,

$$\delta_E(0) = 0 \tag{58}$$

$$\frac{\partial\delta_E(0)}{\partial x} = 0 \tag{59}$$

These conditions look like the usual cantilever boundary conditions for fixed coordinates but are actually specified by the movement of the coordinate system that is attached to the missile and tangent to it at the midpoint. These midpoint conditions can be easily satisfied for a finite difference calculation and need not be used as boundary conditions.

Calculations of special solutions that assume a single-frequency harmonic time variation and involve ordinary differential equations must satisfy these midpoint conditions. In this paper, these solutions are calculated by pairs of differential equations in x for the rod's aft section ($x_1 \le x \le 0$) and its fore section ($0 \le x \le x_2$). Equations (53), (54), (58), and (59) are boundary conditions for the aft part and Eqs. (55), (56), (58), and (59) are boundary conditions for the fore part. This use of six boundary conditions implies discontinuities in $\partial^2\delta_E/\partial x^2$ and $\partial^3\delta_E/\partial x^3$ at the central junction point. These six boundary conditions for the ordinary differential equations in x are not, however, independent, and it can be shown that Eqs. (53) and (54) imply Eqs. (55) and (56) and vice versa.[13] Thus, the calculations will not have discontinuities in shear force and moment at the junction point.

Bent Missile

If the rod were inelastically deformed on launch, its shape would be represented by the sum of a fixed deformation rotating with the missile and an elastic deformation:

$$\delta_E = \tilde{\delta}_E(x,t) + \delta_{EB}(x)e^{i\phi} \tag{60}$$

where

$$\delta_{EB}(0) = \frac{d\delta_{EB}(0)}{dx} = 0, \qquad p = \dot{\phi}$$

The local inclination of the rod and the lateral location of the c.m. become

$$\Gamma = \tilde{\Gamma} + \Gamma_B e^{i\phi} \tag{61}$$

$$\delta_c = \tilde{\delta}_c + \delta_{cB}e^{i\phi} \tag{62}$$

where

$$\tilde{\Gamma} = \frac{\partial\tilde{\delta}_E}{\partial x} \qquad \Gamma_B = \frac{d\delta_{EB}}{dx}$$

$$\tilde{\delta}_c = \left(\frac{1}{L}\right)\int_{x_1}^{x_2}\tilde{\delta}_E\, dx \qquad \delta_{cB} = \left(\frac{1}{L}\right)\int_{x_1}^{x_2}\delta_{EB}\, dx$$

The location of the centerline of the disks relative to the projectile c.m. is

$$\delta = \tilde{\delta}_E - \tilde{\delta}_c + \delta_B e^{i\phi} \tag{63}$$

where

$$\delta_B = \delta_{EB} - \delta_{cB}$$

For the rigid aerodynamic structure extension,

$$\delta_{EB}(x) = \delta_{EB}(x_1) + (x - x_1)\Gamma_B(x_1) \qquad x_0 \le x \le x_1$$

$$\delta_{EB}(x) = \delta_{EB}(x_2) + (x - x_2)\Gamma_B(x_2) \qquad x_2 \le x \le x_3 \tag{64}$$

Now Γ_B produces an aerodynamic force and moment that rotates with the projectile. If the fins are bent with respect to the rod, an additional differential force that rotates with the missile is produced and must be added to Eq. (26),

$$dF_{\mathrm{BF}} = g_1 c_{f1}\Gamma_{\mathrm{BF}}(x)e^{i\phi}\, dx \tag{65}$$

where $\Gamma_{\mathrm{BF}}(x)$ is a measure of bent fin damage. If the fin damage extends beyond the rear end of the rod, it will exert an additional force and moment on the rod at $x = x_1$. This bent fin force term also has to be included in the quadratic roll moment appearing in Eq. (43),

$$g_x\dot{p} + R\{\dot{J}_7 - \dot{Q}\bar{J}_6 - g_2(J_9 + J_{9\mathrm{BF}})\} = g_2(C_\ell)_{\mathrm{linear}} \tag{66}$$

where

$$J_{9\mathrm{BF}} = -ie^{i\phi}\int_{x_0}^{0} c_{f1}\Gamma_{\mathrm{BF}}\bar{\delta}\, dx$$

Equation (35) for the c.m. motion of the elastic missile, Eq. (37) for the aerodynamic moment, and Eq. (23) for the missile's angular motion become

$$(V/d)(\dot{\xi} - iQ) = -g_2\left(C_{L\alpha}\xi - C_{N\,\mathrm{BF}}e^{i\phi}\right) + N \tag{67}$$

$$M = -i(g_1 d)\lfloor C_{M\alpha}\xi + (C_{M\dot{\alpha}} + C_{Mq})(\dot{\xi}d/V) - J_3(t) - \dot{J}_4(t)(d/V) - J_5(t) - C_{M\,\mathrm{BF}}e^{i\phi}\rfloor \tag{68}$$

$$I_t\dot{Q} - ipI_xQ = -i(g_1 d)\lfloor C_{M\alpha}\xi + (C_{M\dot{\alpha}} + C_{Mq})(\dot{\xi}d/V) - C_{M\,\mathrm{BF}}e^{i\phi}\rfloor - iJ_E - imd^2 J_6 \tag{69}$$

where $C_{N\,\mathrm{BF}}$ and $C_{M\,\mathrm{BF}}$ are defined in Appendix B.

In the flexing motion equation (51), δ_E is replaced by $\tilde{\delta}_E$ in the elastic and damping force terms and the bent fins terms added to F and $\hat{F}_a$ to obtain the partial differential equation for a bent missile,

$$\frac{\partial^2\tilde{\delta}_E}{\partial t^2} + \omega_0^2\frac{\partial^4\tilde{\delta}_E}{\partial x^4} + \omega_0^2\hat{k}\tilde{c}_{d4} - g_2 L\left[c_{f1}\Gamma + \frac{\partial}{\partial t}(c_{f2}\Gamma - c_{f1}\delta_E)\left(\frac{d}{V}\right) - \frac{\partial(c_D\delta)}{\partial x}\right] = E_1\xi + E_2\dot{\xi} + ix\dot{Q} - N - E_{\mathrm{BF}}e^{i\phi} \tag{70}$$

In the boundary conditions specified by Eqs. (53–59), δ_E is replaced by $\tilde{\delta}_E$ in the partial derivatives on the left side of the equations. The conditions at x_1 are modified by force or moment contributions from any fin damage between x_0 and x_1 that may have occurred.

Two Special Solutions

The general solution for the motion of a bent missile requires the numerical integration of one partial differential equation [Eq. (70)] and three ordinary differential equations [Eqs. (66), (67), and (69)]. Two special solutions for constant spin involve only ordinary differential equations and can be computed on a personal computer.

The first special solution is that for the steady-state motion of a bent missile with a constant spin. This motion is called trim motion and has the form

$$\xi = \xi_T e^{ipt} \tag{71}$$

$$\tilde{\delta}_E = \tilde{\delta}_{ET}e^{ipt} \tag{72}$$

$$\delta_E = (\delta_{EB} + \tilde{\delta}_{ET})e^{ipt} = \delta_{ET}e^{ipt} \tag{73}$$

For this motion, beam damping is zero ($\tilde{c}_{dr} = 0$), and $\xi_T(p)$ and $\delta_{ET}(p, x)$ for specific values of spin can be computed from a fourth-order inhomogeneous differential equation.[13]

For trim motion, $\dot{p} = \dot{J}_7 = 0$, $\dot{Q} = ip^2\xi_T$, and Eq. (66) becomes a simple equality of two functions of p:

$$f_2(p) = f_1(p) \tag{74}$$

Where

$$f_1 = p - p_{ss}, \qquad f_2 = -R\left\{i\left(p^2/g_2\right)\xi_T\bar{J}_{6T} + J_{9T}\right\}(C_{\ell p}d/V)^{-1}$$

Equilibrium values of spin are determined by the intersection of these two curves. These equilibrium spins locate spin lock-in possibilities. The stability of an equilibrium spin can be found by the integration of Eqs. (66), (67), and (69–70) for nonconstant spin near an equilibrium spin.

The second special solution is a harmonic transient solution to the homogeneous equation. The general solution for constant spin can be represented as the linear combination of the trim solution and an infinite series of the transient solutions.

A rigid symmetric finned missile flying with constant spin has two natural frequencies $\dot{\phi}_{1R}$ and $\dot{\phi}_{2R}$, where $\dot{\phi}_{1R} \cong -\dot{\phi}_{2R}$. Each of the flexure frequencies given by Eq. (47) would give rise to two coning frequencies $\pm\omega_K$. The frequencies present in the motion of an elastic projectile would form an infinite sequence, where the first two would be related to $\dot{\phi}_{1R}$ and $\dot{\phi}_{2R}$, and the later ones would evolve from $\pm\omega_K$, that is, $\dot{\phi}_{2K+1} \cong \omega_K$ and $\dot{\phi}_{2K+2} \cong -\omega_K$. Transient solutions of Eqs. (38) and (52) have the form

$$\xi = T_k e^{A_k t} \tag{75}$$

$$\delta_E = \psi_k(x) T_k e^{A_k t} \tag{76}$$

where

$$A_k = \lambda_k + i\dot{\phi}_k, \qquad T_k = K_{k0} e^{i\phi_{k0}}$$

A fourth-order ordinary differential combined with a simple algebraic equation and appropriate boundary conditions can be numerically solved[13] to yield A_k and $\psi_k(x)$.

Induced Drag

The aerodynamic drag force is directed along the velocity vector and the linear normal force on a disk is perpendicular to the disk axis, which is canted at an angle of $\xi - \Gamma$ with respect to the velocity vector. Thus, the normal force has a quadratic contribution to the drag. This contribution is called induced drag and can be computed for a pitching and flexing missile,

$$g_1 C_{DI} = -\int_{x_0}^{x_3} \left[(\beta - \Gamma_y) \frac{\partial F_y}{\partial x} + (\alpha - \Gamma_z) \frac{\partial F_z}{\partial x} \right] \mathrm{d}x \tag{77}$$

Although the nonlinear axial flow along the rod has a quadratic contribution to drag, this contribution is usually significantly less than the induced drag.

The normal force distribution will be approximated by its dominant term $g_1 c_{f1}(\xi - \Gamma)$,

$$C_{DI} = \int_{x_0}^{x_3} c_{f1}(\xi - \Gamma)(\bar{\xi} - \bar{\Gamma})\,\mathrm{d}x = C_{N\alpha}|\xi|^2 - \bar{\xi} J_1 - \xi \bar{J}_1 + J_D \tag{78}$$

where

$$J_D = \int_{x_0}^{x_3} c_{f1} \Gamma \bar{\Gamma}\,\mathrm{d}x$$

The first term in Eq. (78) is the well-known expression for the induced drag of a rigid missile. The induced drag for a particular angular mode is

$$C_{DIk} = C_k K_{k0}^2 \tag{79}$$

where

$$C_k = C_{N\alpha} - J_{1k} - \bar{J}_{1k} + J_{Dk}, \qquad J_{1k} = \int_{x_0}^{x_3} c_{f1} \psi_k \,\mathrm{d}x$$

$$J_{Dk} = \int_{x_0}^{x_3} c_{f1} \frac{\mathrm{d}\psi_k}{\mathrm{d}x} \frac{\mathrm{d}\bar{\psi}_k}{\mathrm{d}x}\,\mathrm{d}x$$

Numerical Results

The theory of this paper has been applied to a fin stabilized 20-caliber rod flying at 6000 ft/s. The finned missile has a 1-caliber nose extension and a 1-caliber fin extension (Fig. 3). The necessary parameters for these missiles are given in Appendix C.

A measure of the flight flexibility of a missile is the ratio of the first elastic frequency to the rigid-missile aerodynamic frequency, $\sigma = \omega_1/\omega_R$. The equations for the transient solution have been solved for the first natural frequency $\dot{\phi}_1$, and the results are plotted vs σ in Fig. 4. This paper's theory is compared with the three-body theory and we see that it has a similar variation but predicts lower frequencies. Note that the frequency for $\sigma = 5$ is near 0.6 times the rigid-missile value. The first two flexing frequencies $\dot{\phi}_3/\omega_1$ and $\dot{\phi}_5/\omega_2$ are plotted vs σ in Fig. 5. For small σ, both frequencies are slightly larger than the free–free beam values, but they approach these values as σ grows.

The forward end of the rod bends away from the velocity vector for the aerodynamic frequency ($\psi_{12} \leq 0$), whereas it bends toward the velocity vector for the first two flexing frequencies. The magnitude of the forward end motion is plotted against σ in Fig. 6. At $\sigma = 5$, we see that 0.1 rad of angular motion causes 0.18 diam of flexing

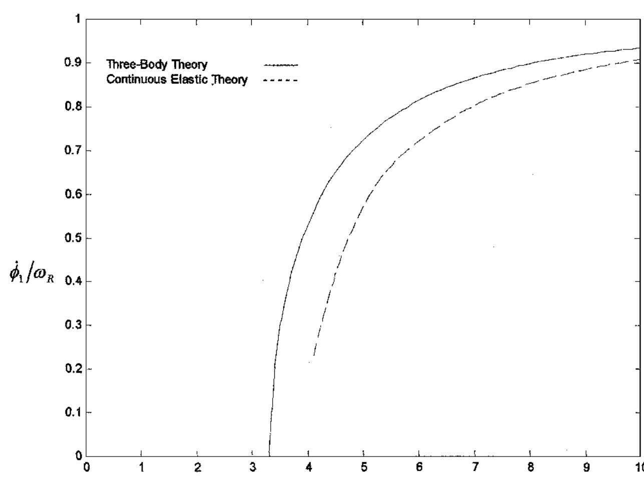

Fig. 4 For finned missile, $\dot{\phi}_1/\omega_R$ vs σ.

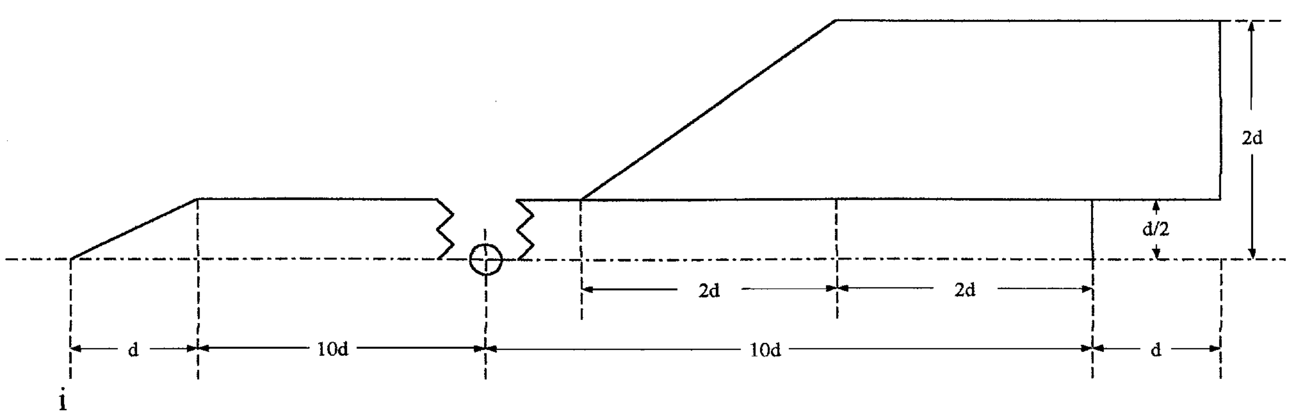

Fig. 3 Finned missile.

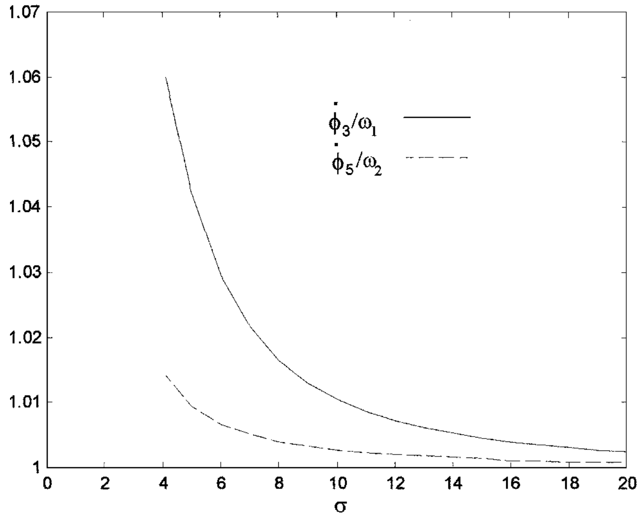

Fig. 5 For finned missile, $\dot{\phi}_3/\omega_1$ and $\dot{\phi}_5/\omega_2$ vs σ.

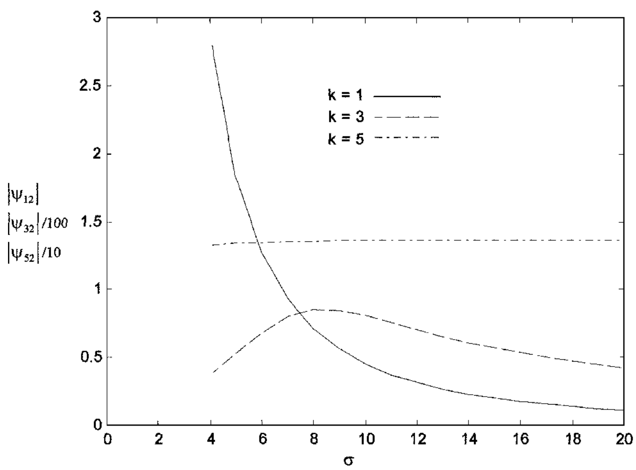

Fig. 6 For k = 1, 3, and 5, $|\psi_{k2}|$ vs σ.

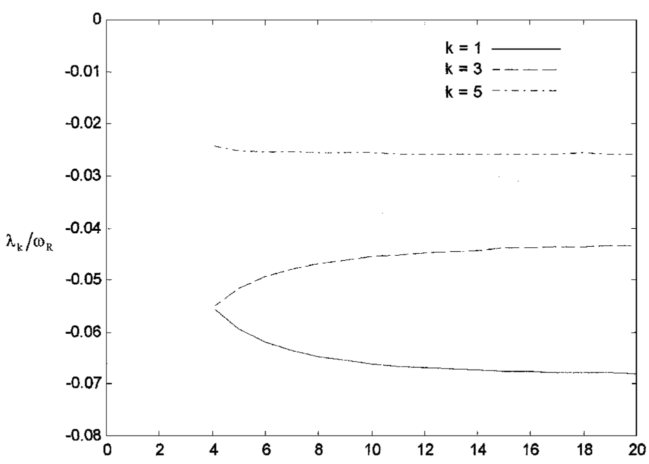

Fig. 7 For k = 1, 3, and 5, λ_k/ω_R vs σ.

motion. For the flexing frequencies one diameter of flexing motion is associated with 0.018 rad (1.0 deg) of angular motion for the first flexing mode and 0.072 rad (4.1 deg) of angular motion for the second flexing mode.

The damping rates of the three modes for no beam damping ($\hat{k} = 0$) are plotted vs σ in Fig. 7. The damping rate for the first flexing mode is 30% less than that for the aerodynamic mode, whereas the damping rate for the second flexing mode is 55% less than that for the aerodynamic mode. In Fig. 8 λ_1/ω_R is plotted vs p/ω_R for

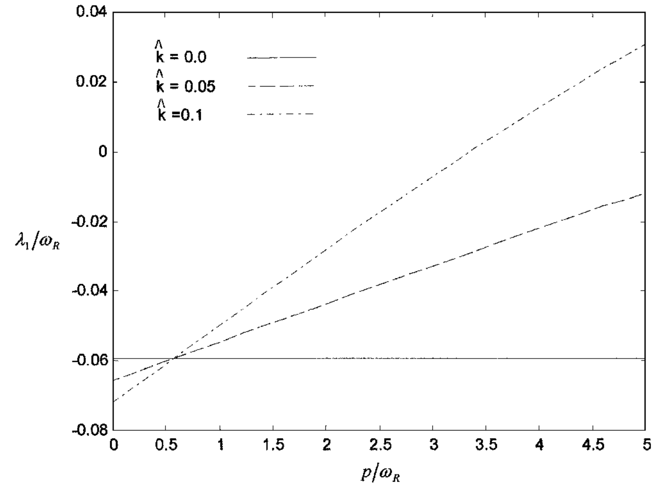

Fig. 8 For σ = 5 and $\hat{k}$ = 0, 0.05, and 0.10, λ_1/ω_R vs p/ω_R.

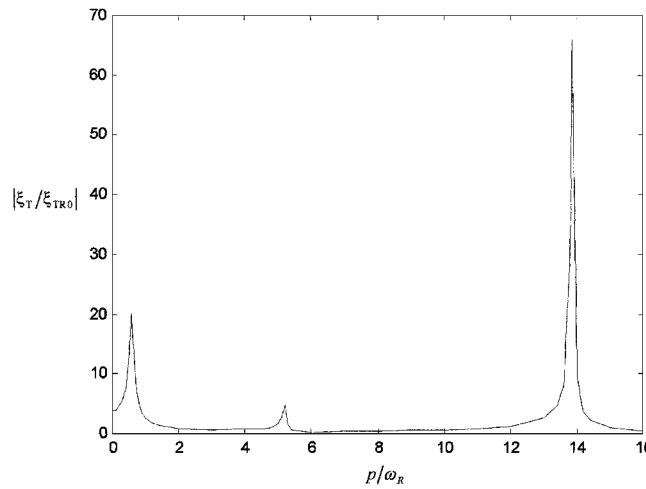

Fig. 9 For bent finned missile at σ = 5, $|\xi_T/\xi_{TR0}\|$ vs p/ω_R.

two nonzero values of beam damping and $\sigma = 5$. For this value of σ, $\dot{\phi}_1/\omega_R$ is 0.6, and beam damping reduces the damping rate when p/ω_R exceeds this value. The behavior was first observed by Platus[1] and predicted by the three-body theory.[5]

The induced drag coefficient for the aerodynamic mode is 12% less than that for a rigid missile at $\sigma = 5$. Although the nose of the elastic missile bends to a larger angle of attack, the fins bend to a lesser angle of attack and the net effect is less induced drag. The induced drag coefficients for the two flexing modes have been calculated, and it was found that for the same amplitude motion the induced drag for the symmetric mode is twice that for the antisymmetric mode. Because both ends of the missile bend to a lesser angle of attack for the antisymmetric mode, this mode should have much smaller induced drag.

For the bent missile calculations we will assume no fin damage ($\Gamma_{BF} = 0$) and the bent rod will be described by a pair of quartic curves:

$$\begin{aligned} \delta_{EB} &= d_{11}x^2 + d_{21}x^4 \qquad -10 \le x \le 0 \\ &= d_{12}x^2 + d_{22}x^4 \qquad 0 \le x \le 10 \end{aligned} \tag{80}$$

We will assume the rear of the rod is undeformed ($d_{11} = d_{21} = 0$) and the values of d_{12} and d_{22} are given in Appendix C.

In Fig. 9, the magnitude of the ratio of the trim angle of attack to the zero spin trim angle of a rigid missile is plotted vs spin for $\sigma = 5$. The zero spin trim angle of the elastic missile is about four times that of a rigid missile. The first resonant peak is near $p/\omega_R = 0.6$ and shows a fivefold magnification of the zero spin elastic trim. A smaller resonant peak occurs at the elastic frequency near 5 and much larger

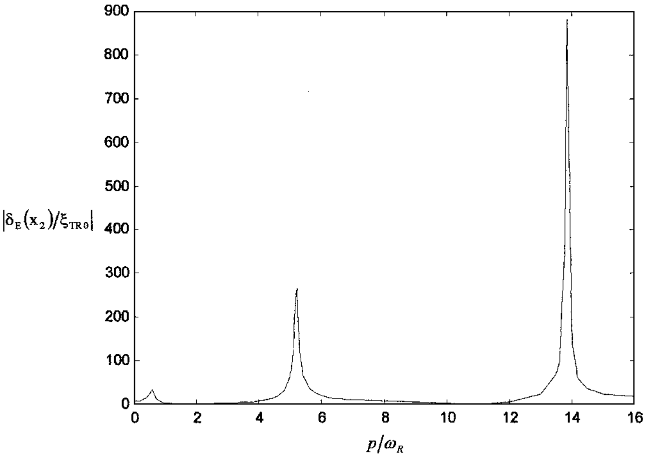

Fig. 10 For bent finned missile at $\sigma = 5$, $|\delta_E(x_2)/\xi_{TR0}|$ vs p/ω_R.

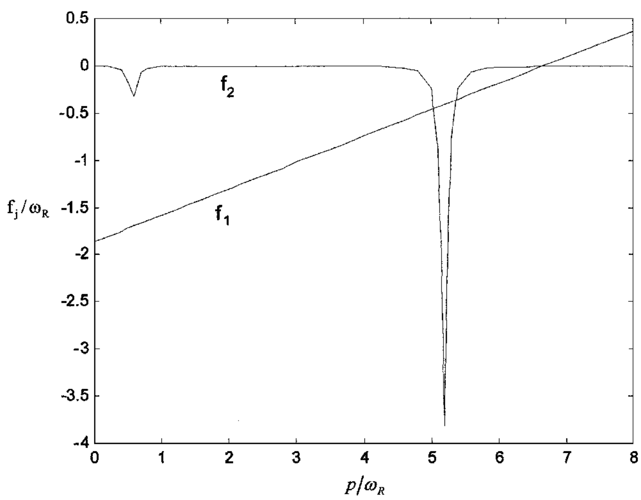

Fig. 11 For bent finned missile at $\sigma = 5$, f_j/ω_R vs p/ω_R.

resonant peak near 14. The magnitude of the rod deflection is plotted vs p/ω_R in Fig. 10. The trim deflection for the first elastic mode is about eight times that for the aerodynamic mode, whereas the trim deflection for the second elastic mode is three times that for the first elastic mode.

In Fig. 11, f_2 is plotted vs p/ω_R. A sample f_1 line for $p_{ss}/\omega_R = 6.5$ is also shown. The intersections of these curves are loci of possible spin–yaw lock-ins. The actual occurrence of lock-in would be shown by integration of the partial differential equation for flexing motion. The three-body theory of Ref. 5 showed the first and third intersections to be locations of spin–yaw lock-in.

Conclusions

The free-flight motion of a symmetric elastic missile can be described by two second-order ordinary differential equations and one fourth-order partial differential equation. The resulting motion contains two aerodynamic frequencies and an infinity of elastic frequencies. The presence of elasticity causes a reduction in the aerodynamic frequencies as the stiffness decreases. The presence of aerodynamic loads causes the elastic frequencies to increase as stiffness decreases.

The elastic modes are damped by the aerodynamic loads but at a lesser rate than the aerodynamic modes. Beam damping damps the aerodynamic mode when spin is less than the aerodynamic frequency and undamps it when the spin exceeds it. The induced drag for the aerodynamic mode is less for an elastic missile than for a rigid missile.

The trim angular motion and its associated flexing motion have resonant amplification when spin is near a transient frequency, and pairs of spin equilibriums can occur at each frequency. The three-body theory indicates one of each pair can cause spin–yaw lock-in.

Appendix A: Functions

$$J_1(t) = \int_{x_0}^{x_3} c_{f1}\Gamma\,\mathrm{d}x, \qquad J_2(t) = \int_{x_0}^{x_3} (c_{f2}\Gamma - c_{f1}\delta_E)\,\mathrm{d}x$$

$$J_3(t) = \int_{x_0}^{x_3} c_{f1}\Gamma x\,\mathrm{d}x, \qquad J_4(t) = \int_{x_0}^{x_3} (c_{f2}\Gamma - c_{f1}\delta_E)x\,\mathrm{d}x$$

$$J_5(t) = \int_{x_0}^{x_3} c_D\delta\,\mathrm{d}x + \delta(x_1)c_{D\,\mathrm{bp}} \qquad J_6(t) = \left(\frac{1}{L}\right)\int_{x_1}^{x_2} x\delta_E\,\mathrm{d}x$$

$$J_{6T}(p,t) = \left(\frac{1}{L}\right)\int_{x_1}^{x_2} x\delta_{ET}\,\mathrm{d}x$$

$$J_7(t) = -\left(\frac{i}{L}\right)\int_{x_1}^{x_2} (\dot{\delta}\bar{\delta} + \dot{\Gamma}_m\bar{\Gamma})\,\mathrm{d}x$$

$$J_8(t) = \left(\frac{1}{16L}\right)\int_{x_1}^{x_2} (\dot{\Gamma} - 2ip\Gamma)\,\mathrm{d}x$$

$$J_9(t) = i\int_{x_0}^{x_3} c_{f1}\left[(\xi - \Gamma) + (\dot{\delta}_E - ixQ)\left(\frac{d}{V}\right)\right]\bar{\delta}\,\mathrm{d}x$$

$$J_{9\mathrm{BF}}(t) = ie^{i\phi}\int_{x_0}^{x_3} c_{f1}\Gamma_{\mathrm{BF}}\bar{\delta}\,\mathrm{d}x$$

$$J_{9T}(p,t) = i\int_{x_0}^{x_3} c_{f1}\left[(\xi_T - \Gamma_T - \Gamma_{\mathrm{BF}}) + (\delta_{ET} - x\xi_T)\left(\frac{pd}{V}\right)\right]\bar{\delta}_T\,\mathrm{d}x$$

$$J_E(t) = -(g_1 d)\left[J_3 + \dot{J}_4\left(\frac{d}{V}\right) + J_5\right]$$

$$J_N(t) = [I_t\dot{N} - (ipI_x)N]\left(\frac{d}{V}\right)$$

$$\tilde{c}_{dr} = 2\omega_1^{-1}\frac{\partial^r}{\partial x^r}\left(\frac{\partial\tilde{\delta}_E}{\partial t} - ip\tilde{\delta}_E\right)$$

$$N(t) = g_2\left[J_1 + \dot{J}_2\left(\frac{d}{V}\right)\right] - \ddot{\delta}_c, \qquad E_1(x) = g_2(C_{N\alpha} - Lc_{f1})$$

$$E_2(x) = g_2[C_{N\dot{\alpha}} + C_{Nq} - L(2c_{f2} - xc_{f1})]\left(\frac{d}{V}\right)$$

$$E_{\mathrm{BF}} = g_2(C_{N\,\mathrm{BF}} - Lc_{f1}\Gamma_{\mathrm{BF}})$$

$$\dot{\Gamma}_m(x,t) = \frac{(\dot{\Gamma} - 2ip\Gamma + iQ)}{16}$$

$$f_1(t) = I_1[\xi(t) - \Gamma(x_1,t)] + \lfloor I_{11}\dot{\xi}(t) + I_1\dot{\delta}_E(x_1,t) + (I_3 - I_5)\dot{\Gamma}(x_1,t)\rfloor\left(\frac{d}{V}\right)$$

$$f_2(t) = I_2[\xi(t) - \Gamma(x_2,t)] + [I_{12}\dot{\xi}(t) + I_2\dot{\delta}_E(x_2,t) + (I_4 - I_6)\dot{\Gamma}(x_2,t)]\left(\frac{d}{V}\right)$$

$$m_1(t) = I_3[\xi(t) - \Gamma(x_1,t)] + [I_{13}\dot{\xi}(t) + I_3\dot{\delta}_E(x_1,t) + (I_9 - I_7)\dot{\Gamma}(x_1,t)]\left(\frac{d}{V}\right) - m_{1D}$$

$$m_2(t) = I_4[\xi(t) - \Gamma(x_2,t)] + [I_{14}\dot{\xi}(t) + I_4\dot{\delta}_E(x_2,t) + (I_{10} - I_8)\dot{\Gamma}(x_2,t)]\left(\frac{d}{V}\right) - m_{2D}$$

$$m_{1D}(t) = I_{1D}[\delta_E(x_1, t) - \delta_c(t)] + I_{3D}\Gamma(x_1, t)$$

$$m_{2D}(t) = I_{2D}[\delta_E(x_2, t) - \delta_c(t)] + I_{4D}\Gamma(x_2, t)$$

Appendix B: Aerodynamic Integrals and Boundary Conditions Integrals

$$C_{N\alpha} = \int_{x_0}^{x_3} c_{f1}\,\mathrm{d}x, \qquad C_{M\alpha} = \int_{x_0}^{x_3} c_{f1}x\,\mathrm{d}x$$

$$C_{Nq} = \int_{x_0}^{x_3} (c_{f2} - xc_{f1})\,\mathrm{d}x, \qquad C_{Mq} = \int_{x_0}^{x_3} (c_{f2} - xc_{f1})x\,\mathrm{d}x$$

$$C_{N\dot\alpha} = \int_{x_0}^{x_3} c_{f2}\,\mathrm{d}x, \qquad C_{M\dot\alpha} = \int_{x_0}^{x_3} c_{f2}x\,\mathrm{d}x$$

$$C_{N\,\mathrm{BF}} = \int_{x_0}^{x_3} c_{f1}\Gamma_{\mathrm{BF}}\,\mathrm{d}x, \qquad C_{M\,\mathrm{BF}} = \int_{x_0}^{x_3} c_{f1}\Gamma_{\mathrm{BF}}x\,\mathrm{d}x$$

$$I_1 = \int_{x_0}^{x_1} c_{f1}\,\mathrm{d}x, \qquad I_2 = \int_{x_2}^{x_3} c_{f1}\,\mathrm{d}x$$

$$I_3 = \int_{x_0}^{x_1} (x - x_1)c_{f1}\,\mathrm{d}x, \qquad I_4 = \int_{x_2}^{x_3} (x - x_2)c_{f1}\,\mathrm{d}x$$

$$I_5 = \int_{x_0}^{x_1} c_{f2}\,\mathrm{d}x, \qquad I_6 = \int_{x_2}^{x_3} c_{f2}\,\mathrm{d}x$$

$$I_7 = \int_{x_0}^{x_1} (x - x_1)c_{f2}\,\mathrm{d}x, \qquad I_8 = \int_{x_2}^{x_3} (x - x_2)c_{f2}\,\mathrm{d}x$$

$$I_9 = \int_{x_0}^{x_1} (x - x_1)^2c_{f1}\,\mathrm{d}x, \qquad I_{10} = \int_{x_2}^{x_3} (x - x_2)^2c_{f1}\,\mathrm{d}x$$

$$I_{11} = 2I_5 - I_3 - x_1I_1, \qquad I_{12} = 2I_6 - I_4 - x_2I_2$$

$$I_{13} = 2I_7 - I_9 - x_1I_3, \qquad I_{14} = 2I_8 - I_{10} - x_2I_4$$

$$I_{1D} = \int_{x_0}^{x_1} c_D\,\mathrm{d}x + c_{D\,\mathrm{bp}}, \qquad I_{2D} = \int_{x_2}^{x_3} c_D\,\mathrm{d}x$$

$$I_{3D} = \int_{x_0}^{x_1} (x - x_1)c_D\,\mathrm{d}x, \qquad I_{4D} = \int_{x_2}^{x_3} (x - x_2)c_D\,\mathrm{d}x$$

$$I_{1\mathrm{BF}} = \int_{x_0}^{x_1} c_{f1}\Gamma_{\mathrm{BF}}\,\mathrm{d}x, \qquad I_{3\mathrm{BF}} = \int_{x_0}^{x_1} (x - x_1)c_{f1}\Gamma_{\mathrm{BF}}\,\mathrm{d}x$$

Appendix C: Finned Missile Parameters

$$L = 20 \qquad V = 1830\ \mathrm{m/s}$$

$$d = 0.107\ \mathrm{m} \qquad \rho = 1.03\ \mathrm{kg/m^3}$$

$$m = 51.1\ \mathrm{kg} \qquad x_{01} = x_{23} = 1$$

$$I_x = 0.073\ \mathrm{kg\text{-}m^2} \qquad a_s = 335\ \mathrm{m/s}$$

$$I_t = 19.47\ \mathrm{kg\text{-}m^2} \qquad a_L = 2\left[\sqrt{(V/a_s)^2 - 1}\right]^{-1}$$

$$\begin{aligned} c_{f1} &= 4(11 - x) && 10 < x \le 11 \\ &= e^{2(x-10)} && -5 < x \le 10 \\ &= -(2/\pi)(15 + 3x)a_L && -7 < x \le -5 \\ &= (12/\pi)a_L && -11 \le x \le -7 \end{aligned}$$

$$\begin{aligned} c_{f2} &= 2(11 - x)^2 && 10 < x \le 11 \\ &= 2 + 0.5\left(1 - e^{2(x-10)}\right) && -5 < x \le 10 \\ &= 2.5 + (1/3\pi)(15 + 3x)^2a_L && -7 < x \le -5 \\ &= 2.5 - (12/\pi)(6 + x)a_L && -11 \le x \le -7 \end{aligned}$$

$$\begin{aligned} c_D &= (0.30)(11 - x) - 0.295e^{-10(x-10)} && 10 < x \le 11 \\ &= 0.0050 && -5 < x \le 10 \\ &= 0.0050 - 0.025(5 + x) && -7 < x \le -5 \\ &= 0.0100 && -11 \le x \le -7 \end{aligned}$$

$$c_{D\,\mathrm{bp}} = 0.14 \qquad C_{\ell p} = -18$$

$$d_{12} = 10^{-3} \qquad d_{22} = -0.25 \times 10^{-5}$$

$$C_{N\alpha} = 9.7 \qquad C_{M\alpha} = -34.4$$

$$C_{Mq} = -980 \qquad C_{M\dot\alpha} = -190$$

References

[1] Platus, D. H., "Aeroelastic Stability of Slender, Spinning Missiles," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 1, 1992, pp. 144–151.

[2] Legner, H. H., Lo, E. Y., and Reinecke, W. G., "On the Trajectory of Hypersonic Projectiles Undergoing Geometry Changes," AIAA Paper 94-0719, Jan. 1994.

[3] Mikhail, A. G., "In-Flight Flexure and Spin Lock-in for Antitank Kinetic Energy Projectiles," *Journal of Spacecraft and Rockets*, Vol. 33, No. 5, 1996, pp. 657–664; also U.S. Army Research Lab., Rept. ARL TR–882, Aberdeen Proving Grounds, MD, Oct. 1995.

[4] Heddadj, S., Cayzac, R., and Renard, J., "Aeroelasticity of High L/D Supersonic Bodies: Theoretical and Numerical Approach," AIAA Paper 2000-0390, Jan. 2000.

[5] Murphy, C. H., and Mermagen, W. H., "Flight Mechanics of an Elastic Symmetric Projectile," *Journal of Guidance, Control, and Dynamics*, Vol. 24, No. 6, 2001; also U.S. Army Research Lab., Rept. ARL TR-2255, AD A390525, Aberdeen Proving Grounds, MD, April 2000.

[6] Murphy, C. H., "Some Special Cases of Spin–Yaw Lock-In," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 6, 1989, pp. 771–776; also U.S. Army Ballistic Research Lab., Rept. BRL MR 3609, AD A185295, Aberdeen Proving Grounds, MD, Aug. 1987.

[7] Wood, R. M., and Murphy, C. H., "Aerodynamic Derivatives for Both Steady and Nonsteady Motion of Slender Missiles," *Journal of the Aerospace Sciences*, Vol. 22, No. 12, 1955, pp. 670, 671; also U.S. Army Ballistic Research Lab., Rept. BRL MR 880, AD 66178, Aberdeen Proving Grounds, MD, April 1955.

[8] Munk, M. M., "The Aerodynamic Forces on Airship Hulls," NACA TR 184, 1923.

[9] Devan, L., and Mason, L. A., "Aerodynamics of Tactical Weapons to Mach Number 8 and Angle of Attack 180: Part II Computer Program and Users Guide," Naval Surface Weapons Center, NSWC-TR-81-358, Dahlgren, VA, Sept. 1981, URL: http://stinet.dtic.mil [cited 4 February 2002].

[10] Murphy, C. H., "Free Flight Motion of Symmetric Missiles," U.S. Army Ballistic Research Lab., BRL Rept. 1216, AD 442757, Aberdeen Proving Grounds, MD, July 1963, URL: http://stinet.dtic.mil [cited 4 February 2002].

[11] deSilver, C. W., *Vibration Fundamentals and Practice*, CRC Press, New York, 2000, p. 351.

[12] Geradin, M., and Rixen, D., *Mechanical Vibrations: Theory and Applications to Structural Dynamics*, Wiley, Chichester, England, U.K., 1994, pp. 179–201.

[13] Murphy, C. H., and Mermagen, W. H., "Flight Motion of a Continuously Elastic Finned Missile," U.S. Army Research Lab., TR-2754, AD 403 876, Aberdeen Proving Grounds, MD, June 2002, URL: http://stinet.dtic.mil [cited 4 February 2002].